Gamete Motion Algorithm

Position Update: The current position vector $\mathbf{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is updated at each time step (Δt)

by the formula

$$\mathbf{P}^{new} = \mathbf{P} + (\Delta t) \times speed \times \mathbf{V}^{new},$$

where $speed$ depends on the specific gamete in question. The new velocity unit vector $\mathbf{V}^{new}$ is computed from random perturbations of the current velocity vector as determined below.

Coordinate Transformations: It is simpler to apply the required velocity perturbations in a coordinate system where the current velocity vector points entirely in the z' direction, i.e.

$$\mathbf{V}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The transformation required is produced by applying a rotation θ about the z-axis and ϕ about the x-axis as follows:

$$\mathbf{V}' = \begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

After inverting we have

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{bmatrix} \cos\theta & -\cos\phi\sin\theta & \sin\phi\sin\theta \\ \sin\theta & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and simplifying gives

$$\begin{pmatrix} \sin\phi\sin\theta \\ -\sin\phi\cos\theta \\ \cos\phi \end{pmatrix} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}.$$

With $S = \sqrt{V_x^2 + V_y^2}$, we can write

$$\sin\theta = \frac{V_x}{S} \quad \text{and} \quad \sin\phi = S$$

$$\cos\theta = \frac{-V_y}{S} \quad \text{and} \quad \cos\phi = V_z \ .$$

If we denote the new velocity vector in the transformed plane as

$$\mathbf{V}'_{new} = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix}$$

then rotation back into the original coordinate system gives

$$\mathbf{V}^{new} = \begin{bmatrix} \cos\theta & -\cos\phi\sin\theta & \sin\phi\sin\theta \\ \sin\theta & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix}$$

or

$$\mathbf{V}^{new} = \begin{pmatrix} V_x^{new} \\ V_y^{new} \\ V_z^{new} \end{pmatrix} = \frac{1}{S} \begin{pmatrix} R_z V_x S - R_y V_x V_z - R_x V_y \\ R_z V_y S - R_y V_y V_z + R_x V_x \\ R_z V_z S + R_y S^2 \end{pmatrix} .$$

Note that if $S = 0$ there is no need to rotate back, so

$$\mathbf{V}^{new} = \begin{pmatrix} V_x^{new} \\ V_y^{new} \\ V_z^{new} \end{pmatrix} = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} .$$

Non-Phototactic Motion: At the next time step we want the gamete to move along a unit vector which is tilted by a slight random angle α $(-\text{limit} < \alpha < \text{limit})$ from the z'- axis, and rotated in the $x'y'$- plane by the random angle $0 \le \beta \le \pi$ (β measured from x$'$- axis), i.e.

$$\mathbf{V}'_{new} = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = \begin{pmatrix} \cos\beta\sin\alpha \\ \sin\beta\sin\alpha \\ \cos\alpha \end{pmatrix}$$

Phototactic Motion: In this case we add a small increment $\delta\,(0 < \delta < \text{deltaLimit})$ to V_z and renormalize by

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \frac{1}{D} \begin{pmatrix} V_x \\ V_y \\ \delta + V_z \end{pmatrix}$$

where $D = \sqrt{1 + (\delta + V_z)^2 - V_z^2}$, and then apply the non-phototactic motion method as given above.